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2690 [April, 1918]. Proposed by E. V. HUNTINGTON, Harvard University.

Find the maximum value of

$$y = \frac{\sin x \cos (x + \varphi)}{\cos (x + \theta) \sin (x + \beta + \theta)}. \tag{1}$$

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Taking the logarithm of both sides, differentiating, and setting the result equal to zero we get

$$\cot x - \tan (x + \varphi) + \tan (x + \theta) - \cot (x + \beta + \theta) = 0 \tag{2}$$

and from the two forms of this

$$\cot x - \cot (x + \beta + \theta) = \tan (x + \varphi) - \tan (x + \theta)$$

and

$$\cot x + \tan (x + \theta) = \cot (x + \beta + \theta) + \tan (x + \varphi)$$

we get

$$\frac{\sin (\beta + \theta)}{\sin (\varphi - \theta)} = \frac{\sin x \sin (x + \beta + \theta)}{\cos (x + \varphi) \cos (x + \theta)} \tag{3}$$

and

$$\frac{\cos (x + \varphi)}{\cos (x + \theta)} = \frac{\cos (\beta + \theta - \varphi)}{\cos \theta} \cdot \frac{\sin x}{\sin (x + \beta + \theta)}. \tag{4}$$

By (3) and (4), we have

$$\sqrt{\frac{\cos (\beta + \theta - \varphi) \sin (\varphi - \theta)}{\cos \theta \sin (\beta + \theta)}} = \frac{\cos (x + \varphi)}{\sin x} = \cot x \cos \varphi - \sin \varphi.$$

Hence,

$$\cot x = \tan \varphi + \sec \varphi \sqrt{\frac{\cos (\beta + \theta - \varphi) \sin (\varphi - \theta)}{\cos \theta \sin (\beta + \theta)}}. \tag{5}$$

(4) and (1) give, where y_1 is the maximum value of y,

$$y_1 = \frac{\cos (\beta + \theta - \varphi)}{\cos \theta} \cdot \frac{\sin^2 x}{\sin^2 (x + \beta + \theta)};$$

whence

$$\frac{1}{\sqrt{y_1}} = \sqrt{\frac{\cos \theta}{\cos (\beta + \theta - \varphi)}} \{\cos (\beta + \theta) + \sin (\beta + \theta) \cot x\}$$

and eliminating $\cot x$ by (5), we have

$$y_1 = \cos^2 \varphi / \{\sqrt{\cos \theta \cos (\beta + \theta - \varphi)} + \sqrt{\sin (\varphi - \theta) \sin (\beta + \theta)}\}^2.$$

As in 2689 we may prove that y_1 is a maximum as found for the case $\beta = \varphi$ from which we may infer from the manner that β enters into the expression that it is a maximum for all values of β.

2692 [April, 1918]. Proposed by J. L. RILEY, Stephenville, Texas.

A cube is cut at random by a plane, what is the chance that the section is a hexagon?

SOLUTION BY C. F. GUMMER, Queen's University, Kingston, Ont.

Let the faces of the cube be $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$. Let the plane be

$$lx + my + nz = p,$$

where $n = \cos \theta > 0$, $l = \sin \theta \cos \phi$, $m = \sin \theta \sin \phi$. The normal from the origin is equally likely to have any direction, and, when the direction is fixed, equally likely to have any length. Since $\sin \theta d\theta d\phi$ is an element of the solid angle traced by the normal, and dp an element of its length, the relative probability of the plane lying within given limits is $\iiint \sin \theta dp d\theta d\phi$. On account of the symmetry of the cube, it may be assumed that the normal through O to the plane meets the surface of the cube within the triangle (0, 0, 1), (1, 0, 1), and (1, 1, 1); so that $n > l > m > 0$, that is, $0 < \phi < \pi/4$, $0 < \theta < \tan^{-1} \sec \phi$.

The distances from the plane, with regard to sign, of the vertices $(1, 1, 1)$, $(1, -1, 1)$, $(-1, 1, 1)$, $(1, 1, -1)$, $(-1, -1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, -1)$ are $l + m + n - p$, $l - m + n - p$, etc.; and these are in descending order, except the two middle ones, which partake of the property only when $l + m > n$, that is when $\theta > \tan^{-1} (\cos \phi + \sin \phi)^{-1}$.

The plane will cut the cube when $l + m + n > p > - l - m - n$, and will cut it in a hexagon when $l + m - n > p > - l - m + n$. Hence, the required chance is

$$\frac{\displaystyle\int_0^{\pi/4} \int_{\tan^{-1}(\cos \phi+\sin \phi)^{-1}}^{\tan^{-1} \sec \phi} \int_{\cos \theta-\sin \theta(\cos \phi+\sin \phi)}^{-\cos \theta+\sin \theta(\cos \phi+\sin \phi)} \sin \theta \, dp \, d\theta \, d\phi}{\displaystyle\int_0^{\pi/4} \int_0^{\tan^{-1} \sec \phi} \int_{-\cos \theta-\sin \theta(\cos \phi+\sin \phi)}^{\cos \theta+\sin \theta(\cos \phi+\sin \phi)} \sin \theta \, dp \, d\theta \, d\phi}.$$

The integrations for p and θ having been effected, the inverse tangents in ϕ may be removed by integration by parts, after which the substitution $u = \tan \phi$ reduces the integrands to rational functions of u. The resulting value for the probability is $2/\sqrt{3} - (4\sqrt{2}/\pi) \tan^{-1} 1/\sqrt{2}$, or $.0465\cdots$.

2693 [April, 1918]. Proposed by W. F. HARLOW, Portland, Oregon.

A cow is tethered with a rope, length l, to a peg on the opposite side of a wall, height h, the peg being at a distance a from the wall. Find the area over which the cow can graze.

SOLUTION BY C. F. GUMMER, Queen's University, Kingston, Ont.

Let the bottom of the wall be $x = z = 0$; the top $x = 0$, $z = h$; and the peg $(- a, 0, 0)$. The cow being at $(x, y, 0)$, the lengths of the two parts of the rope being u and v, and the point at which the rope crosses the wall being $(0, t, h)$, we must have, if the rope is taut,

$$u^2 = a^2 + h^2 + t^2, \qquad v^2 = x^2 + (t - y)^2 + h^2,$$

and $u + v = l$.

The solution of these equations in t, u, v is

$$t = \frac{yp \pm lr}{2(l^2 - y^2)}, \qquad u = \frac{lp \pm yr}{2(l^2 - y^2)}, \qquad v = \frac{lq \mp yr}{2(l^2 - y^2)},$$

where $p = a^2 + l^2 - x^2 - y^2$, $q = l^2 + x^2 - a^2 - y^2$, and $r = \sqrt{p^2 - 4(a^2 + h^2)(l^2 - y^2)}$.

We require the conditions that u and v may be positive and t real.

From the obtuse triangle $(- a, 0, 0)$, $(0, 0, h)$, (x, y, h) it is clearly necessary that

$$(1) \qquad l^2 > a^2 + h^2 + x^2 + y^2.$$

Other necessary conditions are that $p^2 \geqq 4(a^2 + h^2)(l^2 - y^2)$, and (since $l > |y|$ and $|p| > r$) that $p > 0$. From these two it follows that

$$(2) \qquad x^2 \leqq l^2 + a^2 - y^2 - 2\sqrt{a^2 + h^2} \sqrt{l^2 - y^2},$$

which again implies that $(\sqrt{l^2 - y^2} - \sqrt{a^2 + h^2})^2 \geqq h^2$, that is (with the help of (1)) that

$$(3) \qquad y^2 \leqq l^2 - (\sqrt{a^2 + h^2} + h)^2.$$

It remains to prove that (2) and (3) are not only necessary but sufficient. For this we need only observe that they make r real (and positive) $l > |y|$, $p > r$, and q positive. The lower signs will then give a suitable solution in t, u, and v.

The area to be computed is therefore

$$A = 2 \int_0^{\sqrt{l^2-(\sqrt{a^2+h^2}+h)^2}} \sqrt{a^2 + l^2 - y^2 - 2\sqrt{a^2 + h^2}\sqrt{l^2 - y^2}}\cdot dy.$$

Writing $\sqrt{l^2 - y^2} = u$, $\sqrt{a^2 + h^2} = c$, we get

$$A = 2 \int_{c+h}^{l} \frac{u\sqrt{(u - c)^2 - h^2}}{\sqrt{l^2 - u^2}} \, du.$$

To reduce to standard elliptic integrals, we may make

$$u = \frac{\lambda v + \mu}{v + 1},$$